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Lefschetz invariants and Young characters for representations of the hyperoctahedral groups

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Abstract

The ring $R(B_n)$ of virtual $\mathbb{C}$-characters of the hyperoctahedral group B_n has two $\mathbb{Z}$-bases consisting of permutation characters, and the ring structure associated with each basis of them defines a partial Burnside ring of which $R(B_n)$ is a homomorphic image. In particular, the concept of Young characters of B_n arises from a certain set $\mathcal{U}_n$ of subgroups of B_n, and the $\mathbb{Z}$-basis of $R(B_n)$ consisting of Young characters, which is presented by L. Geissinger and D. Kinch [7], forces $R(B_n)$ to be isomorphic to a partial Burnside ring $\Omega(B_n,\mathcal{U}_n)$. The linear $\mathbb{C}$-characters of B_n are analyzed with reduced Lefschetz invariants which characterize the unit group of $\Omega(B_n,\mathcal{U}_n)$. The parabolic Burnside ring $\mathcal{PB}(B_n)$ is a subring of $\Omega(B_n,\mathcal{U}_n)$, and the unit group of $\mathcal{PB}(B_n)$ is isomorphic to the four group. The unit group of the parabolic Burnside ring of the even-signed permutation group D_n is also isomorphic to the four group.

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1 Introduction

Let G be a finite group, and let G-**set** be the category of finite left G-sets and G-equivariant maps. The Burnside ring $\Omega(G)$, which is the Grothendieck ring of the category G-**set**, is the commutative unital ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes $[X]$ of finite left G-sets X with disjoint union for addition and cartesian product for multiplication. We denote by $R(G)$ the ring of virtual $\mathbb{C}$-characters of G. Set $[n] = \{1, 2, \ldots, n\}$, and let S_n be the symmetric group on $[n]$. We denote by $\mathcal{Y}_n$ the set of Young subgroups of S_n, which is closed under intersection and conjugation. By [15, §7], $\Omega(S_n)$ possesses the partial Burnside ring $\Omega(S_n, \mathcal{Y}_n)$ relative to the Young subgroups as a subring, and $\Omega(S_n, \mathcal{Y}_n) \cong R(S_n)$. This fact means that the characters $1_Y^{S_n}$ induced from the trivial characters 1_Y of Y for $Y \in \mathcal{Y}_n$ form a $\mathbb{Z}$-basis of $R(S_n)$ (see, *e.g.*, [2, Proposition 3]). Let C_2 be a cyclic group of order 2, and let V_n be the direct product $C_2^{(n)}$ of n copies of C_2. We denote by B_n the hyperoctahedral group, that is, the wreath product $C_2 \wr S_n$ defined to be a semidirect product $V_n \rtimes S_n$ of V_n with S_n. Let $\mathcal{Z}_n$ be the set of all products KY of $K \le V_n$ and $Y \in \mathcal{Y}_n$ with $|V_n : K| \le 2$ and $Y \le N_{S_n}(K)$. We establish in §3 that $R(B_n)$ is a homomorphic image of the partial Burnside ring $\Omega(B_n, \widetilde{\mathcal{Z}}_n)$ relative to the set $\widetilde{\mathcal{Z}}_n$ of intersections of subgroups contained in $\mathcal{Z}_n$.

For a ring R, we denote by $R^\times$ the unit group of R. By [13, Example 2], $R(S_n)^\times$ is isomorphic to the four group. There exists a unit of $\Omega(S_n, \mathcal{Y}_n)$ which enables us to describe the sign character $\mathrm{sgn}_n : S_n \to \mathbb{C}$ as a $\mathbb{Z}$-linear combination of the characters $1_Y^{S_n}$ for $Y \in \mathcal{Y}_n$ (see [2, Corollary 2] and [9, §4]); such a description is called Solomon's formula. The ring $R(B_n)$ includes exactly four linear $\mathbb{C}$-characters, and $R(B_n)^\times$ is generated by the nontrivial linear $\mathbb{C}$-characters and -1_{B_n}. In §4 we identify $R(B_n)^\times$ with a subgroup of $\Omega(B_n, \widetilde{\mathcal{Z}}_n)^\times$, and then describe the linear $\mathbb{C}$-characters of B_n as $\mathbb{Z}$-linear combinations of the characters $1_H^{B_n}$ for $H \in \mathcal{Z}_n$.

There is a set $\mathcal{U}_n$ of subgroups of B_n such that the characters $1_H^{B_n}$ for $H \in \mathcal{U}_n$ form a $\mathbb{Z}$-basis of $R(B_n)$ (cf. [7, Corollary II.4]). In §5 we define the partial Burnside ring $\Omega(B_n, \mathcal{U}_n)$ relative to the Young subgroups of B_n, which is a subring of $\Omega(B_n)$ isomorphic to $R(B_n)$. The parabolic Burnside ring $\mathcal{PB}(B_n)$ (cf. [1, §4]) is a subring of $\Omega(B_n, \mathcal{U}_n)$. By [4, (66.29) Corollary], the sign character $\varepsilon_n : B_n \to \mathbb{C}$ is described as a $\mathbb{Z}$-linear combination of the characters $1_H^{B_n}$ for parabolic subgroups H of B_n, whence $\mathcal{PB}(B_n)$ includes a unit α_n corresponding to $\varepsilon_n : B_n \to \mathbb{C}$. There also is a unit β_n of $\Omega(B_n, \mathcal{U}_n)$ corresponding to a natural extension of $\mathrm{sgn}_n : S_n \to \mathbb{C}$ to B_n such that $\alpha_n\beta_n$ corresponds to the restriction of $\mathrm{sgn}_{2n} : S_{2n} \to \mathbb{C}$ to B_n. By the description of β_n in terms of the characters $1_H^{B_n}$ for $H \in \mathcal{Z}_n \cap \mathcal{U}_n$, we have

$$\beta_n \in \Omega(B_n, \widetilde{\mathcal{Z}}_n)^\times \cap (\Omega(B_n, \mathcal{U}_n)^\times - \mathcal{PB}(B_n)^\times),$$

which proves $\mathcal{PB}(B_n)^\times$ to be isomorphic to the four group.

Let $X \in G$-**set**. To explore the units of $\Omega(G)$, we are mainly concerned with the reduced Lefschetz invariant $\widetilde{\Lambda}_{P(X)}$ of the G-poset $P(X)$ consisting of nonempty and proper subsets of X. The reduced Euler-Poincaré characteristic $\widetilde{\chi}(P(X)^K)$ of the set of K-invariants $P(X)^K$ in $P(X)$ with $K \le G$ is $(-1)^{|K\backslash X|}$, so that $\widetilde{\Lambda}_{P(X)}$ is a unit of $\Omega(G)$ (cf. [11, §5]). As a sequel to this fact, the linear $\mathbb{C}$-characters of B_n are analyzed with reduced Lefschetz invariants which characterize $\Omega(B_n,\mathcal{U}_n)^\times$.

Let D_n be the group of even-signed permutations on $[n]$, which is also a Coxeter group of type D. In §6 we explore the units of the parabolic Burnside ring of D_n.

2 Lefschetz invariant

Following [4, §80], we review the Burnside ring of G and related facts. Let $\mathbf{F}(G)$ be the free abelian group on the set of isomorphism classes of finite left G-sets. Given $X \in G$-**set**, we denote by $\overline{X}$ the isomorphism class of left G-sets including X. Let $\mathbf{F}(G)_0$ be the subgroup of $\mathbf{F}(G)$ generated by the elements $\overline{X_1 \dot\cup X_2} - \overline{X_1} - \overline{X_2}$ for X_1, $X_2 \in G$-**set**. We define a multiplication on the generators of $\mathbf{F}(G)$ by

$$\overline{X_1} \cdot \overline{X_2} = \overline{X_1 \times X_2},$$

where $X_1 \times X_2$ is the cartesian product of X_1 and X_2, and extend it to $\mathbf{F}(G)$ by $\mathbb{Z}$-linearly. Then $\mathbf{F}(G)$ is a commutative unital ring, and $\mathbf{F}(G)_0$ is an ideal of $\mathbf{F}(G)$. We define a commutative unital ring $\Omega(G)$ to be the quotient $\mathbf{F}(G)/\mathbf{F}(G)_0$, and call it the Burnside ring of G. For each $X \in G$-**set**, let $[X]$ be the coset $\overline{X} + \mathbf{F}(G)_0$ of $\mathbf{F}(G)_0$ in $\mathbf{F}(G)$ represented by $\overline{X}$. Then by [4, (80.4) Lemma], $[X_1] = [X_2]$ if and only if $\overline{X_1} = \overline{X_2}$. Hence we may regard $[X]$ as the isomorphism class of left G-sets including $X \in G$-**set**. Multiplication on the generators of $\Omega(G)$ is given by

$$[X_1] \cdot [X_2] = [X_1 \times X_2].$$

Let $\mathrm{C}(G)$ be a full set of non-conjugate subgroups of G. Given $H \le G$, we denote by G/H the set of left cosets gH, $g \in G$, of H in G, and make G/H into a left G-set by defining $d(gH) = dgH$ for all d, $g \in G$. For H, $K \le G$, $G/H \simeq G/K$ if and only if H is a conjugate of K (cf. [4, (80.5) Proposition]). The elements $[G/H]$ for $H \in \mathrm{C}(G)$ form a free $\mathbb{Z}$-basis of $\Omega(G)$. We have

$$[G/H] \cdot [G/U] = \sum_{HgU \in H\backslash G/U} [G/(H \cap {}^gU)] \tag{1}$$

for all H, $U \le G$, where ${}^gU = gUg^{-1}$ (cf. [4, §80 Exercise 2]). The identity of $\Omega(G)$ is $[G/G]$. For shortness' sake, we usually write $1 = [G/G]$.

Let $H \le G$. For each $X \in G$-**set**, we denote by $\mathrm{inv}_H(X)$ or X^H the set of H-invariants in X. There exists a ring homomorphism $\phi_H : \Omega(G) \to \mathbb{Z}$ given by

$$[G/U] \mapsto |\mathrm{inv}_H(G/U)|$$

for all $U \in \mathrm{C}(G)$. For each $X \in G$-**set**, it is obvious that

$$\phi_H([X]) = |X^H|.$$

We set $\widetilde{\Omega}(G) = \prod_{H \in \mathrm{C}(G)} \mathbb{Z}$, and define a map $\phi : \Omega(G) \to \widetilde{\Omega}(G)$ by

$$x \mapsto (\phi_H(x))_{H \in \mathrm{C}(G)}$$

for all $x \in \Omega(G)$. By [4, (80.12) Proposition], this map is a ring monomorphism. We call $\widetilde{\Omega}(G)$ the ghost ring of $\Omega(G)$, and call $\phi : \Omega(G) \to \widetilde{\Omega}(G)$ the Burnside homomorphism or the mark homomorphism. Obviously, $\widetilde{\Omega}(G)^\times = \prod_{H \in \mathrm{C}(G)} \mathbb{Z}^\times$. Hence $\widetilde{\Omega}(G)^\times$ is an elementary abelian 2-group, and so is $\Omega(G)^\times$.

We turn to the concept of (reduced) Lefschetz invariants for finite G-sets. A finite (left) G-set P equipped with order relation $\leq$ is called a finite G-poset if $\leq$ is invariant under the G-action. Let P be a finite G-poset. For each nonnegative integer n, we denote by $Sd_n(P)$ the set of chains $p_0 < p_1 < \cdots < p_n$ of elements of P of cardinality $n+1$, and make $Sd_n(P)$ into a G-set by defining

$$g(p_0 < p_1 < \cdots < p_n) = gp_0 < gp_1 < \cdots < gp_n$$

for all $g \in G$ and $p_0 < p_1 < \cdots < p_n \in Sd_n(P)$. The Lefschetz invariant Λ_P of P and the reduced Lefschetz invariant $\widetilde{\Lambda}_P$ of P are two elements of $\Omega(G)$ given by

$$\Lambda_P = \sum_{i=0}^{\infty} (-1)^i [Sd_i(P)] \quad \text{and} \quad \widetilde{\Lambda}_P = \Lambda_P - 1,$$

respectively, which are introduced by Thévenaz (cf. [3, 11]).

Given $X \in G$-**set**, we denote by $P(X)$ the G-poset consisting of nonempty and proper subsets of X, and explore $\widetilde{\Lambda}_{P(X)}$ from the point of view of combinatorics.

Definition 2.1 Let $X \in G$-**set**. Given $X_0 \in G$-**set**, we define a finite left G-set $\mathrm{Map}(X, X_0)$ to be the set of maps from X to X_0 with the action given by

$$(gf)(x) = gf(g^{-1}x)$$

for all $g \in G$, $f \in \mathrm{Map}(X, X_0)$, and $x \in X$ (cf. [5, §2]). Given a nonnegative integer i and $X_0, X_1, \ldots, X_i \in G$-**set**, we denote by $\mathrm{Map}(X, X_0, X_1, \ldots, X_i)$ the set of all $f \in \mathrm{Map}(X, X_0 \dot\cup X_1 \dot\cup \cdots \dot\cup X_i)$ such that $\mathrm{Im} f \cap X_j \neq \emptyset$ for any $j = 1, 2, \ldots, i$, and make it into a left G-set by defining

$$(gf)(x) = gf(g^{-1}x)$$

for all $g \in G$, $f \in \mathrm{Map}(X, X_0, X_1, \ldots, X_i)$, and $x \in X$.

Lemma 2.2 *Let $X \in G$-**set**. Set $n = |X|$ and $X_1 = \cdots = X_n = G/G$. Then*

$$\widetilde{\Lambda}_{P(X)} = \sum_{i=1}^{n} (-1)^i [\mathrm{Map}(X, \emptyset, X_1, \ldots, X_i)].$$

Proof. Obviously, $[\mathrm{Map}(X, \emptyset, X_1)] = [\mathrm{Map}(X, G/G)] = 1$. We assume that $2 \leq i \leq n$, and define a bijection $\Delta : \mathrm{Map}(X, \emptyset, X_1, \ldots, X_i) \to Sd_{i-2}(P(X))$ by

$$f \mapsto p_0 < p_1 < \cdots < p_{i-2},$$

where

$$p_k = \{x \in X \mid f(x) \in X_j \text{ for some } j \in \{1, 2, \ldots, k+1\}\}$$

for each integer k with $0 \leq k \leq i-2$. Let $g \in G$, and let $f \in \mathrm{Map}(X, \emptyset, X_1, \ldots, X_i)$. We have $(gf)(gx) = f(x)$ for any $x \in X$. Hence, if $\Delta(f) = p_0 < p_1 < \cdots < p_{i-2}$, then $\Delta(gf) = gp_0 < gp_1 < \cdots < gp_{i-2}$. Consequently, we have

$$[\mathrm{Map}(X, \emptyset, X_1)] = 1 \quad \text{and} \quad [\mathrm{Map}(X, \emptyset, X_1, \ldots, X_i)] = [Sd_{i-2}(P(X))]$$

for all integer i with $2 \leq i \leq n$, which implies that

$$\widetilde{\Lambda}_{P(X)} = -1 + \sum_{i=0}^{\infty} (-1)^i [Sd_i(P(X))] = \sum_{i=1}^{n} (-1)^i [\mathrm{Map}(X, \emptyset, X_1, \ldots, X_i)].$$

This completes the proof. $\Box$

By Eq.(1), the set $\Omega(G)^+$ consisting of all elements $\sum_{U \in \mathrm{C}(G)} \ell_U [G/U]$, $\ell_U \geq 0$, of $\Omega(G)$ is an additive semigroup closed under multiplication. We fix $X \in G$-**set**, and define a multiplicative map $\mathrm{Map}(X, -) : \Omega(G)^+ \to \Omega(G)$ by

$$[Y] \mapsto [\mathrm{Map}(X, Y)]$$

for all $Y \in G$-**set**. There exists a unique polynomial map (multiplicative map) $(-)^{[X]} : \Omega(G) \to \Omega(G)$, $y \mapsto y^{[X]}$ extending $\mathrm{Map}(X, -)$ (see [5, §2] and [14, §3]). If $X = X_1 \dot{\cup} X_2$, then $y^{[X]} = y^{[X_1]} \cdot y^{[X_2]}$ for any $y \in \Omega(G)$.

By [14, Lemma 3.6], $\phi((-1)^{[X]}) = ((-1)^{|K \backslash X|})_{K \in \mathrm{C}(G)}$, where $K \backslash X$ is the set of K-orbits in X, and thus $(-1)^{[X]} \in \Omega(G)^\times$. The following proposition is equivalent to [9, Proposition 4.1] and [11, Proposition 5.1].

Proposition 2.3 *For any $X \in G$-**set**, $\widetilde{\Lambda}_{P(X)} = (-1)^{[X]} \in \Omega(G)^\times$.*

We derive Proposition 2.3 from the combinatorial identity

$$(-1)^n = \sum_{i=1}^{n} (-1)^i S(n, i) i!, \tag{2}$$

where $S(n, i)$ is the Stirling number of the second kind (cf. [10, (24d)]). While Eq.(2) is equivalent to [9, Lemma 4.2], the former is nicer than the later for our argument based on entry 3 of the Twelvefold Way (cf. [10, p. 33]).

Proof of Proposition 2.3. Set $n = |X|$ and $X_1 = \cdots = X_n = G/G$. By Lemma 2.2,

$$\widetilde{\Lambda}_{P(X)} = \sum_{i=1}^{n} (-1)^i [\mathrm{Map}(X, \emptyset, X_1, \ldots, X_i)].$$

Let $K \in \mathrm{C}(G)$, and set $m_K = |K \backslash X|$. Then for each integer i with $1 \leq i \leq n$,

$$|\mathrm{Map}(X, \emptyset, X_1, \ldots, X_i)^K| = S(m_K, i)i!,$$

because $S(m_K, i)$ is the number of partitions of an m_K-set into i nonempty subsets. Combining the preceding facts with Eq.(2), we have

$$\phi(\widetilde{\Lambda}_{P(X)}) = \left(\sum_{i=1}^{m_K} (-1)^i S(m_K, i)i! \right)_{K \in \mathrm{C}(G)} = ((-1)^{m_K})_{K \in \mathrm{C}(G)},$$

completing the proof. □

Remark 2.4 For each $X \in G$-**set**, the elements $y^{[X]}$ for $y \in \Omega(G)$, which may be called exponentials, were introduced by A. Dress (cf. [5, §2]), including $(-1)^{[X]}$ (cf. [5, §3]), and the fact that $\phi(\widetilde{\Lambda}_{P(X)}) = ((-1)^{|K \backslash X|})_{K \in \mathrm{C}(G)}$ was generalized in terms of the exponentials (see [12, §6] and [14, §3]).

3 The character ring of B_n

Set $C_2 = \mathbb{Z}^\times$, and let V_n be the direct product $C_2^{(n)}$ of n copies of C_2. The wreath product $B_n := C_2 \wr S_n$ of C_2 with S_n is defined to be the semidirect product

$$V_n \rtimes S_n = \{(x_1, \ldots, x_n)\sigma \mid (x_1, \ldots, x_n) \in V_n \text{ and } \sigma \in S_n\}$$

in which each permutation on $[n]$ acts as an inner automorphism on V_n:

$$\sigma(x_1, \ldots, x_n)\sigma^{-1} = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

If $L \leq V_n$ or if $F \leq S_n$, then we regard L or F as a subgroup of B_n. Given $K \leq V_n$ and $F \leq N_{S_n}(K) := N_{B_n}(K) \cap S_n$, KF is the semidirect product $K \rtimes F$.

Given $J \subset [n]$, we denote by S_J the symmetric group on J, and view it as a subgroup of S_n. For a cycle type $\lambda = (1^{m_1}, \ldots, n^{m_n})$ of a permutation on $[n]$, let S_λ denote a Young subgroup of S_n isomorphic to $S_1^{(m_1)} \times \cdots \times S_n^{(m_n)}$, where each $S_i^{(m_i)}$ is the direct product of m_i copies of S_i.

Let $J \subset [n]$. There exists a linear $\mathbb{C}$-character ϑ_J of V_n given by

$$\vartheta_J((x_1, \ldots, x_n)) = \vartheta(x_1) \cdots \vartheta(x_n) \quad \text{with} \quad \vartheta(x_j) = \begin{cases} x_j & \text{if } j \in J, \\ 1 & \text{otherwise} \end{cases}$$

for all $(x_1, \ldots, x_n) \in V_n$. Set $\overline{J} = [n] - J$. The inertia group $I_{B_n}(\vartheta_J)$ of ϑ_J, which is defined to be $\{a \in B_n \mid \vartheta_J(aba^{-1}) = \vartheta_J(b) \text{ for all } b \in V_n\}$, is

$$V_n(S_J S_{\overline{J}}) = \{(x_1, \ldots, x_n)\sigma \in B_n \mid (x_1, \ldots, x_n) \in V_n \text{ and } \sigma \in S_J S_{\overline{J}}\}$$

(cf. [8, Lemma 25.5]). There exists an extension $\widehat{\vartheta_J}$ of ϑ_J to $I_{B_n}(\vartheta_J)$ given by

$$\widehat{\vartheta_J}((x_1, \ldots, x_n)\sigma) = \vartheta_J((x_1, \ldots, x_n))$$

for all $(x_1, \ldots, x_n) \in V_n$ and $\sigma \in S_J S_{\overline{J}}$. Obviously, $I_{B_n}(\vartheta_J)/V_n \simeq S_J S_{\overline{J}}$. For a $\mathbb{C}$-character ψ of $S_J S_{\overline{J}}$, we denote by $\widehat{\psi}$ the $\mathbb{C}$-character of $I_{B_n}(\vartheta_J)$ given by

$$\widehat{\psi}(g\sigma) = \psi(\sigma)$$

for all $g \in V_n$ and $\sigma \in S_J S_{\overline{J}}$. Set $K_J = \ker \vartheta_J$. Then $S_J S_{\overline{J}} \leq I_{B_n}(\vartheta_J) \leq N_{B_n}(K_J)$.

For each integer i with $0 \leq i \leq n$, we indicate with $[i] \subset [n]$ that $[i]$ is the subset $\{1, 2, \ldots, i\}$ of $[n]$, where $[0]$ is the empty set.

Let $[i] \subset [n]$. We write $\vartheta_i = \vartheta_{[i]}$, $K_i = \ker \vartheta_i$, $S_i = S_{[i]}$, and $S_{\overline{i}} = S_{\overline{[i]}}$ for shortness' sake. Let $\mathrm{Irr}\,(S_i S_{\overline{i}})$ be the set of irreducible $\mathbb{C}$-characters of $S_i S_{\overline{i}}$.

The following proposition is well-known (cf. [7, §II]).

Proposition 3.1 *The irreducible $\mathbb{C}$-characters of B_n consist of the $\mathbb{C}$-characters $(\widehat{\vartheta_i}\widehat{\psi})^{B_n}$ induced from the product $\widehat{\vartheta_i}\widehat{\psi}$ of $\widehat{\vartheta_i}$ and $\widehat{\psi}$ for $[i] \subset [n]$ and $\psi \in \mathrm{Irr}\,(S_i S_{\overline{i}})$.*

Let $J \subset [n]$, and let $\mathcal{P}(J)$ be the set of cycle types of permutations on J. We write $\mathcal{P}(n) = \mathcal{P}([n])$. Recall that for each $\lambda \in \mathcal{P}(J)(= \mathcal{P}(|J|))$, S_λ denotes a Young subgroup of $S_{|J|}$. We set $\mathcal{P}(J, \overline{J}) = \mathcal{P}(J) \times \mathcal{P}(\overline{J})$. Given $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$, let $S_{\lambda_J \lambda_{\overline{J}}}$ denote the product HK of a subgroup H of S_J and a subgroup K of $S_{\overline{J}}$ such that H is a conjugate of S_{λ_J} in S_n and K is a conjugate of $S_{\lambda_{\overline{J}}}$ in S_n.

For each $X \in G$-**set**, let π_X be the permutation character of G which assigns each $g \in G$ the number of fixed elements of X by g, that is, $\pi_X(g) = |X^{\langle g \rangle}|$. For each $H \leq G$, $\pi_{G/H}$ is the character 1_H^G induced from the trivial character 1_H of H.

Theorem 3.2 *The characters $1_{K_i S_{\lambda_i \lambda_{\overline{i}}}}^{B_n}$ induced from the trivial characters $1_{K_i S_{\lambda_i \lambda_{\overline{i}}}}$ of $K_i S_{\lambda_i \lambda_{\overline{i}}}$ for $[i] \subset [n]$ and $(\lambda_i, \lambda_{\overline{i}}) \in \mathcal{P}([i], \overline{[i]})$ form a $\mathbb{Z}$-basis of $R(B_n)$. In particular, the number of irreducible $\mathbb{C}$-characters of B_n is $\sum_{i=0}^{n} |\mathcal{P}([i], \overline{[i]})|$.*

Proof. The second assertion is well-known, and is also an immediate consequence of the first one. Let $J \subset [n]$, and let $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$. If $g \in V_n$ and $\sigma \in S_J S_{\overline{J}}$, then

$$\begin{aligned} g\sigma(h\tau K_J S_{\lambda_J \lambda_{\overline{J}}}) = h\tau K_J S_{\lambda_J \lambda_{\overline{J}}} \quad &\Longleftrightarrow \quad \tau^{-1}h^{-1}(g\sigma)h\tau \in K_J S_{\lambda_J \lambda_{\overline{J}}} \\ &\Longleftrightarrow \quad {}^{\tau^{-1}}(h^{-1}g)\,{}^{\tau^{-1}\sigma}h\tau^{-1}\sigma\tau \in K_J S_{\lambda_J \lambda_{\overline{J}}} \\ &\Longleftrightarrow \quad g\,{}^{\sigma}h \in h\,{}^{\tau}K_J \text{ and } \sigma\tau \in \tau S_{\lambda_J \lambda_{\overline{J}}} \\ &\Longleftrightarrow \quad ghK_J = hK_J \text{ and } \sigma\tau S_{\lambda_J \lambda_{\overline{J}}} = \tau S_{\lambda_J \lambda_{\overline{J}}} \end{aligned}$$

for all $h \in V_n$ and $\tau \in S_J S_{\overline{J}}$, because $\sigma \in N_{S_n}(K_J)$ and $|V_n : K_J| \le 2$, and thus

$$\begin{aligned} 1_{K_J S_{\lambda_J \lambda_{\overline{J}}}}^{I_{B_n}(\vartheta_J)}(g\sigma) &= \pi_{I_{B_n}(\vartheta_J)/(K_J S_{\lambda_J \lambda_{\overline{J}}})}(g\sigma) \\ &= \pi_{V_n/K_J}(g) \cdot \pi_{(S_J S_{\overline{J}})/S_{\lambda_J \lambda_{\overline{J}}}}(\sigma) \\ &= 1_{K_J}^{V_n}(g) \cdot 1_{S_{\lambda_J \lambda_{\overline{J}}}}^{S_J S_{\overline{J}}}(\sigma). \end{aligned}$$

In particular, $1_{V_n S_{\lambda_{\overline{\emptyset}}}}^{I_{B_n}(\vartheta_\emptyset)} = \widehat{1_{S_{\lambda_{\overline{\emptyset}}}}^{S_{\overline{\emptyset}}}}$. Moreover, if $J \ne \emptyset$, then $\vartheta_J = 1_{K_J}^{V_n} - 1_{V_n}$ and

$$(1_{K_J S_{\lambda_J \lambda_{\overline{J}}}}^{I_{B_n}(\vartheta_J)} - \widehat{1_{S_{\lambda_J \lambda_{\overline{J}}}}^{S_J S_{\overline{J}}}})(g\sigma) = (1_{K_J}^{V_n} - 1_{V_n})(g) \cdot 1_{S_{\lambda_J \lambda_{\overline{J}}}}^{S_J S_{\overline{J}}}(\sigma) = (\widehat{\vartheta}_J \widehat{1_{S_{\lambda_J \lambda_{\overline{J}}}}^{S_J S_{\overline{J}}}})(g\sigma)$$

for all $g \in V_n$ and $\sigma \in S_J S_{\overline{J}}$, and consequently,

$$1_{K_J S_{\lambda_J \lambda_{\overline{J}}}}^{B_n} = \widehat{1_{S_{\lambda_J \lambda_{\overline{J}}}}^{S_J S_{\overline{J}}}}^{B_n} + \left(\widehat{\vartheta}_J \widehat{1_{S_{\lambda_J \lambda_{\overline{J}}}}^{S_J S_{\overline{J}}}}\right)^{B_n} = \widehat{1_{S_{\lambda_J \lambda_{\overline{J}}}}^{S_n}} + \left(\widehat{\vartheta}_J \widehat{1_{S_{\lambda_J \lambda_{\overline{J}}}}^{S_J S_{\overline{J}}}}\right)^{B_n}.$$

Let $[i] \subset [n]$. By the above fact with $J = [i]$ and Proposition 3.1, it suffices to verify that the characters $1_{S_{\lambda_i \lambda_{\overline{i}}}}^{S_i S_{\overline{i}}}$ for $(\lambda_i, \lambda_{\overline{i}}) \in \mathcal{P}([i], \overline{[i]})$ form a $\mathbb{Z}$-basis of $R(S_i S_{\overline{i}})$. We identify $S_i S_{\overline{i}}$ and the subgroups $S_{\lambda_i \lambda_{\overline{i}}}$ of $S_i S_{\overline{i}}$ for $(\lambda_i, \lambda_{\overline{i}}) \in \mathcal{P}([i], \overline{[i]})$ with $S_i \times S_{n-i}$ and the subgroups $S_\mu \times S_\nu$ of $S_i \times S_{n-i}$ for $\mu \in \mathcal{P}(i)$ and $\nu \in \mathcal{P}(n-i)$, respectively. By [2, Proposition 3] and [4, §9 Exercise 6], the characters $1_{S_\mu \times S_{n-i}}^{S_i \times S_{n-i}} 1_{S_i \times S_\nu}^{S_i \times S_{n-i}}$ for $\mu \in \mathcal{P}(i)$ and $\nu \in \mathcal{P}(n-i)$ form a $\mathbb{Z}$-basis of $R(S_i \times S_{n-i})$. This, combined with [4, (10.19) Corollary], shows that the characters $1_{S_\mu \times S_\nu}^{S_i \times S_{n-i}}$ for $\mu \in \mathcal{P}(i)$ and $\nu \in \mathcal{P}(n-i)$ form a $\mathbb{Z}$-basis of $R(S_i \times S_{n-i})$, as desired. This completes the proof. □

We quote part of [15, §3] and review the concept of generalized Burnside rings.

Definition 3.3 For a set $\mathcal{D}$ of subgroups of G, we define a $\mathbb{Z}$-lattice $\Omega(G, \mathcal{D})$ to be an additive group consisting of all $\mathbb{Z}$-linear combinations of the elements $[G/H]$ of $\Omega(G)$ for $H \in \mathcal{D}$, and define $\overline{\mathcal{D}} := \{ {}^g H \mid g \in G \text{ and } H \in \mathcal{D}\}$.

The following theorem is a concise version of [15, 3.11 Theorem].

Theorem 3.4 *Let $\mathcal{D}$ be a set of subgroups of G including G, and suppose that*

$$\bigcap_{\langle g \rangle U \le H \in \overline{\mathcal{D}}} H \in \overline{\mathcal{D}}$$

for all $U \in \overline{\mathcal{D}}$ and $g \in N_G(U)$. Then $\Omega(G, \overline{\mathcal{D}})$ has a unique ring structure such that the group homomorphism $\Omega(G, \overline{\mathcal{D}}) \to \prod_{H \in \mathrm{C}(G) \cap \overline{\mathcal{D}}} \mathbb{Z}$ given by

$$x \mapsto (\phi_H(x))_{H \in \mathrm{C}(G) \cap \overline{\mathcal{D}}}$$

for all $x \in \Omega(G, \overline{\mathcal{D}})$ is a ring homomorphism, and the identity of $\Omega(G, \overline{\mathcal{D}})$ is 1. If $\overline{\mathcal{D}}$ is closed under intersection, then $\Omega(G, \overline{\mathcal{D}})$ is a subring of $\Omega(G)$.

We set $\mathcal{X}_n = \{K_J S_{\lambda_J \lambda_{\overline{J}}} \mid J \subset [n] \text{ and } (\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})\}$. Let $\mathcal{Y}_n$ be the set of Young subgroups of S_n, and let $\mathcal{Z}_n$ be the set consisting of all products KY of $K \leq V_n$ and $Y \in \mathcal{Y}_n$ with $|V_n : K| \leq 2$ and $Y \leq N_{S_n}(K)$. We define

$$\widetilde{\mathcal{Z}}_n := \left\{ \bigcap_{H \in \mathcal{S}} H \;\middle|\; \mathcal{S} \in \mathrm{Sub}(\mathcal{Z}_n) \right\},$$

where $\mathrm{Sub}(\mathcal{Z}_n)$ is the set of nonempty subsets of $\mathcal{Z}_n$.

Lemma 3.5 *The following statements hold.*

(a) *The set $\overline{\mathcal{X}_n}$ coincides with $\mathcal{Z}_n$. In particular, $\mathcal{Z}_n$ is closed under conjugation.*

(b) *The set $\widetilde{\mathcal{Z}}_n$ is closed under intersection and conjugation.*

Proof. Suppose that $J \subset [n]$ and $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$. Let $\sigma \in S_n$, and let $g \in V_n$. Then we have ${}^{\sigma}(K_J S_{\lambda_J \lambda_{\overline{J}}}) = K_{\sigma(J)}\, {}^{\sigma}S_{\lambda_J \lambda_{\overline{J}}}$, ${}^{\sigma}S_{\lambda_J \lambda_{\overline{J}}} \in \mathcal{Y}_n$, and ${}^{\sigma}S_{\lambda_J \lambda_{\overline{J}}} \leq N_{S_n}(K_{\sigma(J)})$, where $\sigma(J) = \{\sigma(j) \mid j \in J\}$. Since $\vartheta_J(g\, {}^{\tau}g) = 1$ for any $\tau \in S_J S_{\overline{J}}$, it follows that

$${}^{g}(K_J S_{\lambda_J \lambda_{\overline{J}}}) = \{gh\, {}^{\tau}g\tau \mid h \in K_J \text{ and } \tau \in S_{\lambda_J \lambda_{\overline{J}}}\} = K_J S_{\lambda_J \lambda_{\overline{J}}}.$$

In particular, $\overline{\mathcal{X}_n} \subset \mathcal{Z}_n$. Suppose that $K \leq V_n$ and $Y \in \mathcal{Y}_n$ with $|V_n : K| \leq 2$ and $Y \leq N_{S_n}(K)$. There exists a subset J of $[n]$ such that $K = K_J$. For each $\sigma \in Y$, we have $K_J = {}^{\sigma}(K_J) = K_{\sigma(J)}$, whence $\sigma(J) = J$ and $Y = {}^{\tau}S_{\lambda_J \lambda_{\overline{J}}}$ for some $\tau \in S_J S_{\overline{J}}$ and $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$. This means that KY is a conjugate of $K_J S_{\lambda_J \lambda_{\overline{J}}}$. Consequently, $\overline{\mathcal{X}_n} \supset \mathcal{Z}_n$, and the statement (a) holds. Obviously, $\widetilde{\mathcal{Z}}_n$ is closed under intersection. Hence the statement (b) follows from (a). This completes the proof. □

By Lemma 3.5, $\widetilde{\mathcal{Z}}_n$ satisfies the hypothesis of Theorem 3.4 with $\mathcal{D} = \overline{\mathcal{D}} = \widetilde{\mathcal{Z}}_n$, so that $\Omega(B_n, \widetilde{\mathcal{Z}}_n)$ is a subring of $\Omega(B_n)$ which is called a partial Burnside ring.

We now define a ring homomorphism $\mathrm{char}_G : \Omega(G) \to R(G)$ by

$$[X] \mapsto \pi_X$$

for all $X \in G$-**set** (cf. [14, §6]), and usually write $\mathrm{char} = \mathrm{char}_G$ by omitting subscript G. Given $x \in \Omega(G)$ and $g \in G$, $\mathrm{char}(x)(g) = \phi_{\langle g \rangle}(x)$.

We are successful in finding a natural relationship between $\Omega(B_n, \widetilde{\mathcal{Z}}_n)$ and $R(B_n)$.

Theorem 3.6 *The ring homomorphism* $\mathrm{char} : \Omega(B_n) \to R(B_n)$ *induces an epimorphism from the partial Burnside ring $\Omega(B_n, \widetilde{\mathcal{Z}}_n)$ to $R(B_n)$.*

Proof. The theorem is a consequence of Theorem 3.2. □

4 Units of the character ring of B_n

The set $[n]$ is viewed as a left S_n-set. According to [9, Eq.(3)],

$$\widetilde{\Lambda}_{P([n])} = \sum_{\lambda=(1^{m_1},\ldots,\, n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n} \frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}[S_n/S_\lambda], \tag{3}$$

so that the sign character $\mathrm{sgn}_n : S_n \to \mathbb{C}$ is described as

$$\mathrm{sgn}_n = \sum_{\lambda=(1^{m_1},\ldots,\, n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n+n} \frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}1_{S_\lambda}^{S_n} \tag{4}$$

(see [2, Corollary 2] and [9, Theorem 4.4]). Note that the numbers

$$\frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}$$

for nonnegative integers $m_1,\ldots,\, m_n$ are multinomial coefficients (cf. [10, 1.2]).

Let $\kappa_n : B_n \to \mathbb{C}$ be a linear $\mathbb{C}$-character of B_n given by

$$(x_1,\ldots,\, x_n)\sigma \mapsto \prod_{i=1}^{n} x_i$$

for all $(x_1,\ldots,\, x_n) \in V_n$ and $\sigma \in S_n$. There also exists an extension $\rho_n : B_n \to \mathbb{C}$ of the sign character $\mathrm{sgn}_n : S_n \to \mathbb{C}$ to B_n given by

$$(x_1,\ldots,\, x_n)\sigma \mapsto \mathrm{sgn}_n(\sigma)$$

for all $(x_1,\ldots,\, x_n) \in V_n$ and $\sigma \in S_n$. Let $\varepsilon_n : B_n \to \mathbb{C}$ be the product $\kappa_n\rho_n$ of κ_n and ρ_n, which coincides with the sign character of B_n.

We view the set $\mathbb{Z}^\times = \{1, -1\}$ as a left B_n-set with the action given by

$$(x_1,\ldots,\, x_n)\sigma.x = x \cdot \prod_{i=1}^{n} x_i$$

for all $(x_1,\ldots,\, x_n) \in V_n$, $\sigma \in S_n$, and $x \in \mathbb{Z}^\times$. The set $[n]$ is naturally viewed as a left B_n-set on which V_n acts trivially. Let $[n]^\diamond$ denote the B_n-set $\mathbb{Z}^\times \dot{\cup} [n]$.

Lemma 4.1 *There are exactly three nontrivial linear $\mathbb{C}$-characters $\kappa_n : B_n \to \mathbb{C}$, $\rho_n : B_n \to \mathbb{C}$, and $\varepsilon_n : B_n \to \mathbb{C}$ defined as above in $R(B_n)$, and $\kappa_n(y) = (-1)^{|\langle y\rangle\backslash\mathbb{Z}^\times|}$, $\rho_n(y) = (-1)^{|\langle y\rangle\backslash[n]|+n}$, and $\varepsilon_n(y) = (-1)^{|\langle y\rangle\backslash[n]^\diamond|+n}$ for each $y \in B_n$.*

Proof. By Proposition 3.1, there are exactly three nontrivial linear $\mathbb{C}$-characters of B_n. Let $(x_1,\ldots,\, x_n) \in V_n$, and let $\sigma \in S_n$. Set $y = (x_1,\ldots,\, x_n)\sigma \in B_n$, and assume that σ is a product of pairwise disjoint n_j-cycles σ_j for $j = 1, 2, \dots, r$ with $\sum_j n_j = n$. Obviously, $\kappa_n(y) = (-1)^{|\langle y\rangle\backslash \mathbb{Z}^\times|}$. We have $|\langle y\rangle\backslash[n]| = r$ and

$$|\langle y\rangle\backslash[n]^\diamond| = \begin{cases} r+1 & \text{if } \prod_{i=1}^{n} x_i = -1, \\ r+2 & \text{if } \prod_{i=1}^{n} x_i = 1. \end{cases}$$

Moreover, if $\ell = \sharp\{j \mid n_j \text{ is odd}\}$, then $\rho_n(y) = \mathrm{sgn}(\sigma) = (-1)^{r-\ell} = (-1)^{r+n}$ and $\varepsilon_n(y) = (-1)^{r+n}\prod_{i=1}^{n} x_i$, because $\ell \equiv n \pmod 2$. This completes the proof. □

Lemma 4.2 $R(B_n)^\times = \langle \kappa_n, \eta_n, -1_{B_n}\rangle$.

Proof. The lemma is a consequence of [6, Theorem 5.5.6] (see also Theorem 3.2), [13, Corollary 1.2 and Lemma 2.1], and Lemma 4.1. □

We are now in position to establish the following proposition.

Proposition 4.3 *The nontrivial linear $\mathbb{C}$-characters of B_n are characterized by the reduced Lefschetz invariants. Indeed, $\kappa_n = \mathrm{char}(\widetilde{\Lambda}_{P(\mathbb{Z}^\times)})$, $\rho_n = (-1)^n\mathrm{char}(\widetilde{\Lambda}_{P([n])})$, and $\varepsilon_n = (-1)^n\mathrm{char}(\widetilde{\Lambda}_{P([n]^\diamond)})$. The reduced Lefschetz invariants $\widetilde{\Lambda}_{P(\mathbb{Z}^\times)}$ and $\widetilde{\Lambda}_{P([n])}$, together with -1, generate an elementary abelian subgroup of $\Omega(B_n, \widetilde{\mathcal{Z}}_n)^\times$ isomorphic to $R(B_n)^\times$, and $\widetilde{\Lambda}_{P([n]^\diamond)} = \widetilde{\Lambda}_{P([n])} \cdot \widetilde{\Lambda}_{P(\mathbb{Z}^\times)}$. Moreover,*

$$\begin{aligned}
\widetilde{\Lambda}_{P(\mathbb{Z}^\times)} &= [B_n/(K_nS_n)] - [B_n/B_n], \\
\widetilde{\Lambda}_{P([n])} &= \sum_{\lambda=(1^{m_1},\dots,n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n}\frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}\,[B_n/(V_nS_\lambda)], \\
\widetilde{\Lambda}_{P([n]^\diamond)} &= \sum_{\lambda=(1^{m_1},\dots,n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n}\frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}\,[B_n/(K_nS_\lambda)] \\
&\quad - \sum_{\lambda=(1^{m_1},\dots,n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n}\frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}\,[B_n/(V_nS_\lambda)].
\end{aligned}$$

Proof. The first assertion follows from Proposition 2.3 and Lemma 4.1. We prove the last two assertions. By Lemma 2.2 with $X = \mathbb{Z}^\times$ and $X_1 = X_2 = B_n/B_n$,

$$\widetilde{\Lambda}_{P(\mathbb{Z}^\times)} = -[\mathrm{Map}(\mathbb{Z}^\times, \emptyset, X_1)] + [\mathrm{Map}(\mathbb{Z}^\times, \emptyset, X_1, X_2)] = -[B_n/B_n] + [B_n/(K_nS_n)].$$

We obtain the description of $\widetilde{\Lambda}_{P([n])}$ in a similar fashion to the proof of [9, Eq.(3)]. By Proposition 2.3, $\widetilde{\Lambda}_{P([n]^\diamond)} = \widetilde{\Lambda}_{P([n])} \cdot \widetilde{\Lambda}_{P(\mathbb{Z}^\times)}$, which yields the description of $\widetilde{\Lambda}_{P([n]^\diamond)}$, and the reduced Lefschetz invariants $\widetilde{\Lambda}_{P(\mathbb{Z}^\times)}$, $\widetilde{\Lambda}_{P([n])}$, and $\widetilde{\Lambda}_{P([n]^\diamond)}$ are contained in $\Omega(B_n, \widetilde{\mathcal{Z}}_n)^\times$. Hence it follows from Lemma 4.2 that $\widetilde{\Lambda}_{P(\mathbb{Z}^\times)}$, $\widetilde{\Lambda}_{P([n])}$, and -1 generate an elementary abelian subgroup of $\Omega(B_n, \widetilde{\mathcal{Z}}_n)^\times$ isomorphic to $R(B_n)^\times$. This completes the proof. □

The following descriptions of nontrivial linear $\mathbb{C}$-characters of B_n are obtained; see Eq.(5) in §5 for Solomon's formula of the sign character $\varepsilon_n : B_n \to \mathbb{C}$.

Corollary 4.4

$$\begin{aligned}
\kappa_n &= 1_{K_nS_n}^{B_n} - 1_{B_n},\\
\rho_n &= \sum_{\lambda=(1^{m_1},\ldots,n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n+n}\,\frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}\,1_{V_nS_\lambda}^{B_n},\\
\varepsilon_n &= \sum_{\lambda=(1^{m_1},\ldots,n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n+n}\,\frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}\,1_{K_nS_\lambda}^{B_n}\\
&\quad - \sum_{\lambda=(1^{m_1},\ldots,n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n+n}\,\frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}\,1_{V_nS_\lambda}^{B_n}.
\end{aligned}$$

Proof. The corollary is an immediate consequence of Proposition 4.3. (The formulae of κ_n and ρ_n can also be obtained by a calculation and Eq.(4), respectively.) □

5 The Young subgroups of the hyperoctahedral groups

Given $J \subset [n]$, we define a subgroup L_J of V_n by

$$L_J = \{(x_1, \ldots, x_n) \in V_n \mid x_k = 1 \text{ for all } k \in \overline{J}\}.$$

Let $\mathcal{U}_n$ denote the set of products $L_JS_{\lambda_J\lambda_{\overline{J}}}$ of L_J and $S_{\lambda_J\lambda_{\overline{J}}}$ for $J \subset [n]$ and $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$, and let $\mathcal{E}_n$ denote the set of products $L_J(S_{\lambda_J\lambda_{\overline{J}}}S_J)$ of L_J and $S_{\lambda_J\lambda_{\overline{J}}}S_J$ for $J \subset [n]$ and $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$. Obviously, $\mathcal{E}_n \subset \mathcal{U}_n$.

We call the subgroups $L_JS_{\lambda_J\lambda_{\overline{J}}}$ of B_n and the characters $1_{L_JS_{\lambda_J\lambda_{\overline{J}}}}^{B_n}$ for $J \subset [n]$ and $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$ the Young subgroups and the Young characters, respectively.

The sets $\mathcal{U}_n$ and $\mathcal{E}_n$ are closed under intersection; they are not closed under conjugation, however. Recall that $\overline{\mathcal{D}} = \{\,{}^yH \mid y \in B_n \text{ and } H \in \mathcal{D}\}$ where $\mathcal{D}$ is $\mathcal{U}_n$ or $\mathcal{E}_n$. Given $[i] \subset [n]$ and $\lambda \in \mathcal{P}(i)$, we write $L_{\bar{i}} = L_{\overline{[i]}}$ and $S_\lambda B_{n-i} = L_{\bar{i}}(S_\lambda S_{\bar{i}})$. The set $\overline{\mathcal{E}}_n$ consists of the conjugates of the parabolic subgroups $S_\lambda B_{n-i}$ for $[i] \subset [n]$ and $\lambda \in \mathcal{P}(i)$, and is closed under intersection (cf. [6, Exercise 2.2]). To explore $\overline{\mathcal{U}}_n$, we make $\mathbb{Z}^\times \times [n]$ into a left B_n-set by defining

$$(x_1, x_2, \ldots, x_n)\sigma.(x, i) = (x_{\sigma(i)}x, \sigma(i))$$

for all $(x_1, x_2, \ldots, x_n) \in V_n$, $\sigma \in S_n$, and $(x, i) \in \mathbb{Z}^\times \times [n]$.

Lemma 5.1 *The set $\overline{\mathcal{U}}_n$ is closed under intersection.*

Proof. Suppose that J_1, $J_2 \subset [n]$, $(\lambda_{J_1}, \lambda_{\overline{J_1}}) \in \mathcal{P}(J_1, \overline{J_1})$, $(\lambda_{J_2}, \lambda_{\overline{J_2}}) \in \mathcal{P}(J_2, \overline{J_2})$, $g \in V_n$, and $\sigma \in S_n$. Then ${}^g(L_{\sigma(J_1)}\, {}^\sigma S_{\lambda_{J_1}\lambda_{\overline{J_1}}}) \cap L_{J_2} S_{\lambda_{J_2}\lambda_{\overline{J_2}}}$ is considered to be the intersection of the stabilizers of disjoint subsets

$$N_1^+, \ldots, N_k^+, N_1^-, \ldots, N_k^-, N_{k+1}, \ldots, N_r$$

obtained by a certain partition of $\mathbb{Z}^\times \times [n]$ into nonempty subsets such that

$$N_i^+ = \{g_i.(1, q) \mid q \in Q_i\} \quad \text{and} \quad N_i^- = \{g_i.(-1, q) \mid q \in Q_i\}$$

with $Q_i \subset [n]$ and $g_i \in L_{Q_i}$ for $i = 1, 2, \ldots, k$ and

$$N_i = \{(1, q), (-1, q) \mid q \in Q_i\}$$

with $Q_i \subset [n]$ for $i = k+1, \ldots, r$. Set $g' = g_1 \cdots g_k$ and $J = Q_{k+1} \dot\cup \cdots \dot\cup Q_r$. Then

$$\begin{aligned} {}^{g\sigma}(L_{J_1} S_{\lambda_{J_1}\lambda_{\overline{J_1}}}) \cap L_{J_2} S_{\lambda_{J_2}\lambda_{\overline{J_2}}} &= {}^g(L_{\sigma(J_1)}\, {}^\sigma S_{\lambda_{J_1}\lambda_{\overline{J_1}}}) \cap L_{J_2} S_{\lambda_{J_2}\lambda_{\overline{J_2}}} \\ &= {}^{g'}(L_J\, {}^\tau S_{\lambda_J \lambda_{\overline{J}}}) \\ &= {}^{g'\tau}(L_J S_{\lambda_J \lambda_{\overline{J}}}) \end{aligned}$$

for some $\tau \in S_J S_{\overline{J}}$ and $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$. Consequently, $\overline{\mathcal{U}}_n$ is closed under intersection. This completes the proof. $\Box$

By Lemma 5.1 and [6, Exercise 2.2], $\Omega(B_n, \mathcal{U}_n)$ and $\Omega(B_n, \mathcal{E}_n)$ are subrings of $\Omega(B_n)$ (cf. Theorem 3.4) called partial Burnside rings. The partial Burnside ring $\Omega(B_n, \mathcal{E}_n)$ is known as the parabolic Burnside ring. As for the partial Burnside ring $\Omega(B_n, \mathcal{U}_n)$ relative to the Young subgroups of B_n, we quote [7, Corollary II.4]:

Theorem 5.2 *The characters $1_{L_{\bar{i}} S_{\lambda_i \lambda_{\bar{i}}}}^{B_n}$ induced from the trivial characters $1_{L_{\bar{i}} S_{\lambda_i \lambda_{\bar{i}}}}$ of $L_{\bar{i}} S_{\lambda_i \lambda_{\bar{i}}}$ for $[i] \subset [n]$ and $(\lambda_i, \lambda_{\bar{i}}) \in \mathcal{P}([i], \overline{[i]})$ form a $\mathbb{Z}$-basis of $R(B_n)$.*

Corollary 5.3 *The ring homomorphism* char : $\Omega(B_n) \to R(B_n)$ *induces a ring isomorphism* $\overline{\text{char}} : \Omega(B_n, \mathcal{U}_n) \to R(B_n)$. *In particular, $\Omega(B_n, \mathcal{U}_n)^\times \simeq R(B_n)^\times$.*

Proof. The corollary is a consequence of Theorem 5.2, because $\mathcal{U}_n$ is a set of conjugates of the subgroups $L_{\bar{i}} S_{\lambda_i \lambda_{\bar{i}}}$ for $[i] \subset [n]$ and $(\lambda_i, \lambda_{\bar{i}}) \in \mathcal{P}([i], \overline{[i]})$. $\Box$

The rest of this section is devoted to quite a new view of the units of $\Omega(B_n, \mathcal{U}_n)$.

Proposition 5.4 $|\Omega(B_n, \mathcal{E}_n)^\times| = 4$.

Proof. By [4, (66.29) Corollary] and Corollary 5.3, there is a unique unit α_n of $\Omega(B_n, \mathcal{E}_n)$ such that $\mathrm{char}(\alpha_n) = \varepsilon_n$. Obviously, $-1 \in \Omega(B_n, \mathcal{E}_n)^\times$. Hence we have $|\Omega(B_n, \mathcal{E}_n)^\times| \geq 4$. By Proposition 4.3 and Theorem 5.2, $\widetilde{\Lambda}_{P([n])} \in \Omega(B_n, \mathcal{U}_n)^\times$ and $\widetilde{\Lambda}_{P([n])} \notin \Omega(B_n, \mathcal{E}_n)^\times$. Thus $|\Omega(B_n, \mathcal{U}_n)^\times : \Omega(B_n, \mathcal{E}_n)^\times| \geq 2$. By Lemma 4.1 and Corollary 5.3, we have $|\Omega(B_n, \mathcal{U}_n)^\times| = |R(B_n)^\times| = 8$, whence $|\Omega(B_n, \mathcal{E}_n)^\times| = 4$. This completes the proof. $\square$

We present a technical lemma by which [4, (66.29) Corollary] deduces Eq.(4) and a description of $\varepsilon_n : B_n \to \mathbb{C}$ (see also [6, Propositions 2.3.8 and 2.3.10]):

$$\varepsilon_n = \sum_{i=0}^{n} \sum_{\lambda=(1^{m_1},\ldots,\, i^{m_i}) \in \mathcal{P}(i)} (-1)^{m_1+\cdots+m_i+n} \frac{(m_1+\cdots+m_i)!}{m_1! \cdots m_i!} 1_{S_\lambda B_{n-i}}^{B_n}. \qquad (5)$$

Lemma 5.5 *Let (S_n, X) be the Coxeter system of type A_{n-1}. Given $\lambda \in \mathcal{P}(n)$, let $\mathcal{W}(\lambda)$ be the set of parabolic subgroups W_I of S_n for $I \subset X$ which are conjugates of S_λ. Suppose that $I \subset X$ and $W_I \in W(\lambda)$ with $\lambda = (1^{m_1}, \ldots, n^{m_n}) \in \mathcal{P}(n)$. Then $|I| \equiv m_1 + \cdots + m_n + n \pmod 2$, so that $(-1)^{|I|} = (-1)^{m_1+\cdots+m_n+n}$.*

Proof. We use induction with respect to the partially order $\leq$ on $\mathcal{P}(n)$ given by

$$\mu \leq \nu \quad :\Longleftrightarrow \quad S_\mu \text{ is a conjugate of a subgroup of } S_\nu.$$

If $\lambda = (1^n)$, then $I = \emptyset$, and hence $|I| \equiv 2n \pmod 2$. Assume that $(1^n) < \lambda$. Then $m_k \neq 0$ and $m_{k+1} = \cdots = m_n = 0$ for some $k \in [n]$. We set

$$\mu = \begin{cases} (1^{m_1+2},\, 2^{m_2-1}) & \text{if } k = 2, \\ (1^{m_1+1},\, 2^{m_2}, \ldots,\, (k-1)^{m_{k-1}+1},\, k^{m_k-1},\, 0, \ldots,\, 0) & \text{if } k > 2. \end{cases}$$

Suppose that $I' \subset X$ and $W_{I'} \in W(\mu)$. Then $\mu < \lambda$ and $|I'| = |I| - 1$. By the inductive assumption, $|I'| \equiv m_1 + \cdots + m_n + 1 + n \pmod 2$. Since $|I| = |I'| + 1$, it follows that $|I| \equiv m_1 + \cdots + m_n + n \pmod 2$. This completes the proof. $\square$

What about a unique unit γ_n of $\Omega(B_n, \mathcal{U}_n)$ satisfying $\mathrm{char}(\gamma_n) = \kappa_n$? We are interested in the reduced Lefschetz invariant $\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])}$.

Lemma 5.6 $\kappa_n = \mathrm{char}(\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])})$.

Proof. By Proposition 2.3, $\mathrm{char}(\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])})(y) = (-1)^{|\langle y \rangle \backslash (\mathbb{Z}^\times \times [n])|}$ for all $y \in B_n$. Let $\sigma \in S_n$, and assume that σ is the product of pairwise disjoint n_j-cycles σ_j for $j = 1, 2, \ldots, r$ with $\sum_j n_j = n$. Let $(x_1, \ldots, x_n) \in V_n$, and set $y = (x_1, \ldots, x_n)\sigma$. For each $j \in \{1, 2, \ldots, r\}$, let I_j be the minimal subset of $[n]$ with $\sigma_j \in S_{I_j}$, and set

$$y_j = (x_1^{(j)},\, x_2^{(j)}, \ldots,\, x_n^{(j)})\sigma_j \quad \text{with} \quad x_i^{(j)} = \begin{cases} x_i & \text{if } i \in I_j, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, $y = \prod_{j=1}^{r} y_j$. We now set $s = \sharp\{j \in \{1, 2, \ldots, r\} \mid \prod_{i=1}^{n} x_i^{(j)} = 1\}$, so that $|\langle y\rangle\backslash(\mathbb{Z}^\times \times [n])| = r + s$. Hence it turns out that

$$\kappa_n(y) = \prod_{i=1}^{n} x_i = \prod_{j=1}^{r}\prod_{i=1}^{n} x_i^{(j)} = (-1)^{r-s} = (-1)^{|\langle y\rangle\backslash(\mathbb{Z}^\times \times [n])|}.$$

Consequently, we obtain $\kappa_n = \mathrm{char}(\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])})$, completing the proof. □

The following lemma, which is a basic fact for the left B_n-set $\mathbb{Z}^\times \times [n]$, is crucial.

Lemma 5.7 *Let $\{M_1, \ldots, M_i\}$, i a positive integer, be a partition of $\mathbb{Z}^\times \times [n]$ into nonempty subsets, and view them as elements of the B_n-poset $P(\mathbb{Z}^\times \times [n])$. If each M_j for $j = 1, 2, \ldots, i$ does not include both $(1, q)$ and $(-1, q)$ for any $q \in [n]$, then there exists an element λ of $\mathcal{P}(n)$ such that the intersection of stabilizers of M_j in B_n for $j = 1, 2, \ldots, i$ is a conjugate of S_λ.*

Proof. There is a partition $\{N_1, \ldots, N_k\}$, k a positive integer, of $[n]$ into nonempty subsets such that each M_j for $j = 1, 2, \ldots, i$ consists of either $(1, q)$ or $(-1, q)$, but not both, for each $q \in N_{\ell_1}\dot\cup\cdots\dot\cup N_{\ell_r}$ with $\{N_{\ell_1}, \ldots, N_{\ell_r}\} \subset \{N_1, \ldots, N_k\}$. Let $\widehat{\mathcal{P}}(n)$ be the set of all cycle types to which such partitions $\{N_1, \ldots, N_k\}$ of $[n]$ into nonempty subsets correspond, and take the maximal element μ of $\widehat{\mathcal{P}}(n)$ with respect to the partially order $\leq$ on $\mathcal{P}(n)$ given in the proof of Lemma 5.5. Let $\{N_1, \ldots, N_k\}$ be a partition of $[n]$ into nonempty subsets corresponding to μ which satisfy the above condition. We set $J = N_\ell$, where ℓ is an arbitrary integer with $1 \leq \ell \leq k$. There exists a unique subset Q of J such that

$$J^+ := \{(1, q) \mid q \in Q\}\dot\cup\{(-1, q) \mid q \in J - Q\} \subset M_{j_1}$$

and

$$J^- := \{(1, q) \mid q \in J - Q\}\dot\cup\{(-1, q) \mid q \in Q\} \subset M_{j_2}$$

for some integers j_1 and j_2 with $1 \leq j_1 \neq j_2 \leq i$. Let $g = (x_1, \ldots, x_n) \in L_Q$, and suppose that $x_q = -1$ for all $q \in Q$. Then the stabilizer of J^+ in B_n is ${}^g(L_{\overline{J}} S_J S_{\overline{J}})$, and so is that of J^- in B_n. Observe now that the intersection of stabilizers of M_j for $j = 1, 2, \ldots, i$ in B_n coincides with the intersection of such subgroups of B_n. Hence the assertion is a consequence of Lemma 5.1. This completes the proof. □

Identifying $(-1, q)$ with $n + q \in [2n]$ for all $q \in [n]$, we may consider S_{2n} to be the symmetric group on $\mathbb{Z}^\times \times [n]$. In particular, B_n is viewed as a subgroup of S_{2n}.

Lemma 5.8 *Let $\lambda \in \mathcal{P}(2n)$. Then $B_n \cap {}^\sigma S_\lambda \in \overline{\mathcal{U}}_n$ for all $\sigma \in S_{2n}$, and*

$$[\mathrm{res}_{B_n}^{S_{2n}}(S_{2n}/S_\lambda)] = \sum_{\sigma \in \overline{B_n\backslash S_{2n}/S_\lambda}} [B_n/(B_n \cap {}^\sigma S_\lambda)] \in \Omega(B_n, \mathcal{U}_n),$$

where $\mathrm{res}_{B_n}^{S_{2n}}$ indicates restriction of operators from S_{2n} to B_n and $\overline{B_n\backslash S_{2n}/S_\lambda}$ is a complete set of representatives of double cosets $B_n\sigma S_\lambda$, $\sigma \in S_{2n}$, in S_{2n}.

Proof. Let $\sigma \in S_{2n}$. By Lemma 5.7, $B_n \cap {}^{\sigma}S_\lambda = {}^{g\tau}(L_J S_{\mu_J \mu_{\overline{J}}})$ for some $J \subset [n]$, $g \in L_{\overline{J}}$, $\tau \in S_J S_{\overline{J}}$, and $(\mu_J, \mu_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$. Hence $B_n \cap {}^{\sigma}S_\lambda \in \overline{\mathcal{U}}_n$. The second assertion follows from [4, (80.27) Subgroup Theorem]. This completes the proof. □

There is a formula of the reduced Lefschetz invariant $\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])}$ (cf. Eq.(6)) which is implicit in the proof of a conclusion from the proceeding facts:

Theorem 5.9 *Define three elements α_n, β_n, and γ_n of $\Omega(B_n, \mathcal{U}_n)$ by*

$$\alpha_n = \sum_{i=0}^{n} \sum_{\lambda=(1^{m_1},\ldots,i^{m_i})\in\mathcal{P}(i)} (-1)^{m_1+\cdots+m_i+n} \frac{(m_1+\cdots+m_i)!}{m_1!\cdots m_i!}\,[B_n/(S_\lambda B_{n-i})],$$
$$\beta_n = (-1)^n \widetilde{\Lambda}_{P([n])}, \quad \text{and} \quad \gamma_n = \widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])}.$$

Then $\varepsilon_n = \mathrm{char}(\alpha_n)$, $\rho_n = \mathrm{char}(\beta_n)$, $\kappa_n = \mathrm{char}(\gamma_n)$, and $\alpha_n = (-1)^n \widetilde{\Lambda}_{P([n]\dot\cup(\mathbb{Z}^\times \times [n]))}$. Moreover, $\Omega(B_n, \mathcal{E}_n)^\times = \langle \alpha_n, -1\rangle$, $\Omega(B_n, \mathcal{U}_n)^\times = \langle \beta_n, \gamma_n, -1\rangle$, and $\alpha_n = \beta_n\gamma_n$.

Proof. By Eq.(5), $\varepsilon_n = \mathrm{char}(\alpha_n)$. Obviously, $\alpha_n \in \Omega(B_n, \mathcal{E}_n)$. Since $\alpha_n \neq 1, -1$, it follows from Proposition 5.4 that $\Omega(B_n, \mathcal{E}_n)^\times$ is generated by α_n and -1. By Proposition 4.3 and Lemma 5.6, we have $\rho_n = \mathrm{char}(\beta_n)$, $\beta_n \in \Omega(B_n, \mathcal{U}_n)$, and $\kappa_n = \mathrm{char}(\gamma_n)$. The reduced Lefschetz invariant $\widetilde{\Lambda}_{P([2n])}$ of the left S_{2n}-set $[2n]$ is an element of $\Omega(S_{2n}, \mathcal{Y}_{2n})$ (cf. [9, §4]); for its description, see Eq.(3). We may identify $\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])}$ with $\mathrm{res}^{S_{2n}}_{B_n}(\widetilde{\Lambda}_{P([2n])})$ which is the element of $\Omega(B_n)$ obtained by restriction of operators on S_{2n}-sets appearing in the components of $\widetilde{\Lambda}_{P([2n])}$ from S_{2n} to B_n. By Lemma 5.8, $\mathrm{res}^{S_{2n}}_{B_n}(\widetilde{\Lambda}_{P([2n])}) \in \Omega(B_n, \mathcal{U}_n)$, and thus $\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])} \in \Omega(B_n, \mathcal{U}_n)$. Moreover, it follows from Lemma 4.2 and Corollary 5.3 that $\Omega(B_n, \mathcal{U}_n)^\times$ is generated by β_n, γ_n, and -1. Also, $\alpha_n = \beta_n\gamma_n$, because $\varepsilon_n = \rho_n\kappa_n$. By Proposition 2.3, it turns out that $\alpha_n = (-1)^n \widetilde{\Lambda}_{P([n]\dot\cup(\mathbb{Z}^\times \times [n]))}$. This completes the proof. □

Since $\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])} = \mathrm{res}^{S_{2n}}_{B_n}(\widetilde{\Lambda}_{P([2n])})$, it follows from Eq.(3) and Lemma 5.8 that

$$\widetilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])} = \sum_{\lambda=(1^{m_1},\ldots,(2n)^{m_{2n}})\in\mathcal{P}(2n)} \sum_{\sigma\in \overline{B_n\backslash S_{2n}/S_\lambda}} (-1)^{m_1+\cdots+m_{2n}} \times \frac{(m_1+\cdots+m_{2n})!}{m_1!\cdots m_{2n}!}[B_n/(B_n\cap {}^{\sigma}S_\lambda)]. \tag{6}$$

We close this section with a character theoretical explanation of the formula of κ_n obtained by Eq.(6). For each $\mathbb{C}$-character χ of G, let $\chi|_H$ with $H \le G$ denote the $\mathbb{C}$-character obtained by restriction of χ from G to H.

Lemma 5.10 *Let $\mathbf{M} : G \to GL_n(\mathbb{C})$ be a $\mathbb{C}$-representation of G affording a real valued character χ of G. Then for any $g \in G$,*

$$\det \mathbf{M}(g) = (-1)^{n-\langle \chi|_{\langle g\rangle}, 1_{\langle g\rangle}\rangle},$$

where $\langle \chi|_{\langle g\rangle}, 1_{\langle g\rangle}\rangle$ is the inner product of $\chi|_{\langle g\rangle}$ and $1_{\langle g\rangle}$.

Proof. See the later part of the proof of [14, Theorem A]. □

There is a representation $\mathbf{M}_n : S_n \to GL_n(\mathbb{C})$ given by

$$\sigma \mapsto (\delta_{\sigma^{-1}(i)\,j})_{1\le i,\, j\le n}, \quad \delta \text{ the Kronecker delta,}$$

which affords the permutation character $\pi_{[n]} : S_n \to \mathbb{C}$. Obviously, the sign character $\mathrm{sgn}_n : S_n \to \mathbb{C}$ coincides with the linear $\mathbb{C}$-character $\det \mathbf{M}_n : S_n \to \mathbb{C}$ given by

$$\sigma \mapsto \det \mathbf{M}_n(\sigma)$$

for all $\sigma \in S_n$. Recall that B_n is viewed as a subgroup of S_{2n}. By Lemma 5.10,

$$\det \mathbf{M}_{2n}(\sigma) = (-1)^{\langle \pi_{[2n]}|_{\langle\sigma\rangle}, 1_{\langle\sigma\rangle}\rangle} = (-1)^{|\langle\sigma\rangle\backslash[2n]|}$$

for all $\sigma \in S_{2n}$ (see also [9, Lemma 3.3]). This, combined with Proposition 2.3 and Lemma 5.6, shows that the linear $\mathbb{C}$-character $\det \mathbf{M}_{2n}|_{B_n} : B_n \to \mathbb{C}$ coincides with $\kappa_n : B_n \to \mathbb{C}$. Consequently, we have $\kappa_n = \mathrm{sgn}_{2n}|_{B_n}$. Hence it follows from Eq.(4) and Lemma 5.8 (see also [4, (10.13) Subgroup Theorem]) that

$$\kappa_n = \sum_{\lambda=(1^{m_1},\dots,\,(2n)^{m_{2n}})\in\mathcal{P}(2n)} \ \sum_{\sigma\in \overline{B_n\backslash S_{2n}/S_\lambda}} (-1)^{m_1+\cdots+m_{2n}} \frac{(m_1+\cdots+m_{2n})!}{m_1!\cdots m_{2n}!} 1_{B_n\cap {}^{\sigma}S_\lambda}^{B_n}$$

and $B_n \cap {}^{\sigma}S_\lambda \in \overline{\mathcal{U}}_n$ for all $\lambda \in \mathcal{P}(2n)$ and $\sigma \in S_{2n}$.

6 The parabolic Burnside rings of even-signed permutation groups

We set $D_n = \ker \kappa_n$ and call it the even-signed permutation group on $[n]$. Obviously, $D_n = K_n S_n$, where $K_n = \ker \vartheta_n$. Suppose that $[i] \subset [n]$ and $\lambda \in \mathcal{P}(i)$. We set $S_\lambda D_{n-i} = (K_n \cap L_{\bar{i}}) S_\lambda S_{\bar{i}}$ and set $t = (0, 0, \dots, 1) \in V_n$. Observe that

$$[\mathrm{res}^{B_n}_{D_n}(B_n/(S_\lambda B_{n-i}))] = \begin{cases} [D_n/(S_\lambda D_{n-i})] & \text{if } 0 \le i \le n-1, \\ [D_n/S_\lambda] + [D_n/\,{}^{t}S_\lambda] & \text{if } i = n \end{cases}$$

by [4, (80.27) Subgroup Theorem], which are contained in the parabolic Burnside ring $\mathcal{PB}(D_n)$ (cf. [6, 2.3.11]). We define a map $\mathrm{res}^{B_n}_{D_n} : \mathcal{PB}(B_n) \to \mathcal{PB}(D_n)$ by

$$[B_n/(S_\lambda B_{n-i})] \mapsto [\mathrm{res}^{B_n}_{D_n}(B_n/(S_\lambda B_{n-i}))]$$

for all $[i] \subset [n]$ and $\lambda \in \mathcal{P}(i)$. Set $\alpha'_n = \mathrm{res}^{B_n}_{D_n}(\alpha_n)$ (see Theorem 5.9). Then

$$\begin{aligned} \alpha'_n &= \sum_{i=0}^{n-1} \sum_{\lambda=(1^{m_1},\dots,\,i^{m_i})\in\mathcal{P}(i)} (-1)^{m_1+\cdots+m_i+n} \frac{(m_1+\cdots+m_i)!}{m_1!\cdots m_i!} [D_n/(S_\lambda D_{n-i})] \\ &+ \sum_{\lambda=(1^{m_1},\dots,\,n^{m_n})\in\mathcal{P}(n)} (-1)^{m_1+\cdots+m_n+n} \frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!} ([D_n/S_\lambda] + [D_n/\,{}^{t}S_\lambda]). \end{aligned}$$

Proposition 6.1 $\mathcal{PB}(D_n)^\times = \langle \alpha'_n, -1 \rangle$.

Proof. By the proof of [1, Theorem 4.5], there is an injection from $\mathcal{PB}(D_n)^\times$ to $R(D_n)^\times$ inherited from the ring homomorphism char : $\Omega(D_n) \to R(D_n)$. The sign character $\varepsilon_n|_{D_n} : D_n \to \mathbb{C}$ is the only nontrivial $\mathbb{C}$-character of D_n and $\mathbb{Q}$ is a splitting field for D_n (cf. [6, §5.6]). This, combined with [13, Corollary 1.2 and Lemma 2.1], shows that $R(D_n)^\times$ is isomorphic to the four group. Moreover, by [4, (10.13) Subgroup Theorem] and Eq.(5), we have $\varepsilon_n|_{D_n} = \mathrm{char}(\alpha'_n)$. Consequently, $\mathcal{PB}(D_n)^\times$ is generated by α'_n and -1. This completes the proof. □

Remark 6.2 Let (W, S) be a Coxeter system of type E_6, E_7, or E_8. Then every character of W is rational-valued (cf. [6, 5.3.6]). Moreover, there are exactly two linear $\mathbb{C}$-characters of W (cf. [6, pp. 413–416]). Hence $R(W)^\times$ is isomorphic to the four group and $\mathcal{PB}(W)^\times$ is isomorphic to a subgroup of $R(W)^\times$ (see the proof of Proposition 6.1). Thus it follows from [4, (66.29) Corollary] that $\mathcal{PB}(W)^\times$ is of order 4 and is generated by $\sum_{J \subset S} (-1)^{|J|}[W/W_J]$ and -1, where $W_J = \langle s \mid s \in J \rangle$.

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